FUNCTIONS WHICH HAVE HARMONIC SUPPORT

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1. Introduction. This paper presents some properties of the functions in a certain subclass of the class of subharmonic functions. Consider real valued functions $v(z)$ defined in a domain D of the complex plane.

DEFINITION 1. The function v is in (hs), the class of functions which have harmonic support in D, if and only if v is uniformly bounded on each compact subset of D and for each $z_0 \in D \, \exists h$ harmonic and single valued in $D \ni h(z_0) = v(z_0)$ and $h \leqq v$ in D.

Carathéodory [5] used such functions in constructing a general result on solutions of Dirichlet's problem which are based on convergent sequences of continuous operators. The functions of the class (hs) are continuous subharmonic functions, but Brelot [4, footnote 16] showed the existence of subharmonic functions in C^n which are not in (hs). A simple example of a continuous subharmonic function not in (hs) is the function $v = |z|^\alpha$, $0 < \alpha < 1$, in $|z| < 1$. For suppose $v \in (hs)$ and let h_n denote the support functions at $z_n = 1/n$, $n = 2, 3, \cdots$. The sequence $\{h_n\}$ constitutes a bounded family of harmonic functions which is thus normal. Writing $z = x + iy$, the family $\{\partial h_n / \partial x\}$ is also normal. On the one hand a convergent subsequence will converge uniformly on $|z| \leqq r < 1$, and on the other hand the values at the support points will be that of $\partial v / \partial x = \alpha n^{1-\alpha}$ which tend to infinity for any subsequence. This contradiction implies $v \notin (hs)$. If $1 \leqq \alpha$, then $v \in (hs)$ and indeed any continuous convex function has support planes and is thus in the class (hs). In the remarks to follow we consider convexity in the narrower sense involving continuity.

Montel [9] considered a more general type of convexity. The function $v(x, y)$ is said to be doubly convex if v is convex in x for each fixed y and convex in y for each fixed x. Denote the classes subharmonic, continuous subharmonic, doubly convex and convex respectively as E_0, E_1, E_2 and E_3. He observed that $E_0 \supset E_1 \supset E_2 \supset E_3$ and compared the closure properties under various operations and limiting processes. Clearly the class (hs) lies between E_1 and E_3. However neither (hs) nor E_2 contains the other. The function $v(x, y) = x^2 - y^2$ being itself harmonic belongs to (hs) in the plane but is not in E_2. On the other hand define $v(x, y) = xy$ in quadrants I, II and III and $v(x, y) = 0$ in quadrant IV. One sees readily that v is doubly convex but suppose it has a harmonic support function at an interior point of quadrant IV. In virtue of the maximum principal it must be identically zero but then of course must exceed v in quadrant II. Hence $v \notin (hs)$ in any domain containing

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portions of quadrants II and IV. In the case of functions of a single real variable all five classes are equivalent.

The latter example may be used to illustrate the fact that harmonic support is necessarily a property in the large.

Beckenbach and Reade have shown in a paper to appear that for a linear operator δ satisfying $\delta f=\Delta f$, the Laplacian of f, for $f\in C^2$ and $\delta f\geqq 0$ at a minimum of f then $v\in(hs)$ implies $\delta v\geqq 0$.

The motivation for the study of these functions originated in an attempt to describe the properties of functions $v=\sup_f |f|$ where the functions f are holomorphic and belong to a family $\mathfrak{F}$ which is subuniformly bounded in D. A property is said to be valid subuniformly in D if it is valid uniformly on each compact subset of D, a terminology introduced by G. MacLane. As will be seen these functions are in (hs) and as a matter of fact if D is simply connected they include the subclass of (hs) containing those functions whose logarithm is in (hs).

Initially, certain closure properties will be derived, then an approximation theorem involving infinitely differentiable functions will be proved. From the viewpoint of potential theory a necessary restriction on the class of distribution functions will be obtained. Finally, certain subclasses of (hs) will be discussed.

2. **General properties.** In view of the fact that a family of harmonic functions which is subuniformly bounded above in D is normal in D, the following is equivalent to Definition 1.

Definition 1′. The function v is in (hs) if and only if there is a family $\mathfrak{H}$, subuniformly bounded above, of functions h harmonic and single valued in D such that $v(z)=\sup_h h(z)$.

If there is a function $h\in\mathfrak{H}\ni h(z_0)=v(z_0)$ then h is the support at z_0. Otherwise there is a sequence $\{h_n\}\ni\lim_n h_n(z_0)=v(z_0)$. A convergent subsequence $\{h_{n_i}\}$ will have a limit h which is harmonic, single valued and bounded by v in D. Also $h(z_0)=v(z_0)$ so that h is the support at z_0. Hence Definition 1′ implies Definition 1. The converse is immediate.

Since $h(s)$ is a class of subharmonic functions the maximum principal applies. Specifically let Δ denote a domain and Γ its boundary and suppose $\Delta\cup\Gamma\subset D$. If $v\in(hs)$ in D, h is harmonic in Δ and continuous on $\Delta\cup\Gamma$ and $v\leqq h$ on Γ then $v\leqq h$ on $\Delta\cup\Gamma$. If $v(z_0)=h(z_0)$, $z_0\in\Delta$, then $v=h$ on $\Delta\cup\Gamma$ (cf. Riesz [12, p. 331]). In the latter case we have the additional property that h has a harmonic and single valued continuation in D and is the unique support function for v at any point in Δ. For suppose h^* is the support at z_0 in Δ. Then $h^*(z_0)=v(z_0)=h(z_0)$ and $h^*\leqq v=h$ on $\Delta\cup\Gamma$. The maximum principal implies $h^*=h$ on $\Delta\cup\Gamma$. Thus h^* is the continuation of h in D. The argument also proves its uniqueness.

If u, $v\in(hs)$ and $c\geqq 0$ then $\sup(u, v)$, $u+v$ and cu are in (hs), results which follow readily using either definition.

THEOREM 1. *Let $\mathcal{V}$ denote a family of functions v of class (hs) which is subuniformly bounded above, then $v^*(z)=\sup_v v(z)$ is in (hs).*

Proof. Utilizing Definition 1′, for each $v\in\mathcal{V}$, $\exists$ a family $\mathcal{H}$ of harmonic functions h such that $v(z)=\sup_h h(z)$, $h\in\mathcal{H}$. If $\mathcal{H}^*=\bigcup\mathcal{H}$, then $v^*(z)=\sup_h h(z)$, $h\in\mathcal{H}^*$ implying $v^*\in(hs)$. Proof complete.

The corresponding property is not true in general for subharmonic functions although it is if v^* is upper semicontinuous.

COROLLARY 1.1. *The function $v\in(hs)$ if for each $z_0\in D$ there is a function $u\in(hs)$ such that $u(z_0)=v(z_0)$ and $u\leqq v$ in D.*

Harnack's inequality [6, p. 62] for nonpositive harmonic functions is also valid for nonpositive functions in (hs).

THEOREM 2. *If $v\in(hs)$ in $|z|<1$ and $v\leqq 0$, then*

$$v(0)\,\frac{1+|z|}{1-|z|}\leqq v(z)\leqq v(0)\,\frac{1-|z|}{1+|z|}.$$

Proof. Any support function h satisfies $h\leqq 0$ in $|z|<1$ and hence

$$h(0)\,\frac{1+|z|}{1-|z|}\leqq h(z)\leqq h(0)\,\frac{1-|z|}{1+|z|}.$$

In particular v exceeds its support function at $z=0$ so that

$$v(0)\,\frac{1+|z|}{1-|z|}\leqq v(z).$$

Also $\alpha(1-|z|)/(1+|z|)$ is an increasing function of α so that every support function satisfies

$$h(z)\leqq v(0)\,\frac{1-|z|}{1+|z|}$$

and thus so does $v(z)$. Proof complete.

It may be remarked that the corresponding inequality for nonnegative functions is not valid. It would be however in the class of functions which are the negatives of those in (hs).

Harnack's convergence theorem [6, p. 67] is valid for nonincreasing sequences in (hs).

THEOREM 3. *If $v_n\in(hs)$, $n=1, 2, \cdots$, and $\{v_n\}$ is a nonincreasing sequence, then $v_n\to-\infty$ or $v_n\to v$ subuniformly in D where $v\in(hs)$.*

Proof. Harnack's theorem for subharmonic functions implies $v_n\to-\infty$ or $v_n\to v$ subuniformly in D. In the latter case let h_n denote the support function of v_n at z_0. A convergent subsequence $\{h_{n_i}\}$ will have a limit function h

satisfying $h(z_0)=v(z_0)$. Since $h_{n_i}\leqq v_{n_i}$, we have $h=\lim h_{n_i}\leqq \lim v_{n_i}=v$. Thus $v\in(hs)$. Proof complete.

THEOREM 4. *If $v_n\in(hs)$ and $v_n\to v$ subuniformly in D then $v\in(hs)$.*

Proof. Let $s_n=\sup_k v_k$, $k\geqq n$. Then $s_n\in(hs)$, $n=1, 2, \cdots$, and since $\{s_n\}$ is a nonincreasing sequence with limit v, $v\in(hs)$. Proof complete.

It would be possible at this point using the concept of equicontinuity to prove that a family $\mathcal{V}$ subuniformly bounded above in D is normal in D. Since this result will be found as a consequence of a property of the distribution functions and a theorem of Arsove, we postpone the demonstration. Assuming the result we may make the following comparisons. One might describe the negatives of the functions in (hs) as the functions having harmonic cover. A family of such functions would then be normal if subuniformly bounded below. The only functions belonging to both classes are harmonic functions which indeed have the well known property that a family bounded above or bounded below is normal. A family of subharmonic functions may be bounded above and below and fail to be normal. These remarks give some insight into the different degrees of regularity for the three classes.

3. **An approximation theorem.** The following three lemmas precede the principal result of this section.

LEMMA 1. *Given any positive number ϵ, there exists $\phi(x)$ infinitely differentiable and convex which satisfies*

$$\phi(x) = x \qquad \text{for } \epsilon \leqq x,$$
$$\phi(x) = 0 \qquad \text{for } x \leqq -\epsilon,$$

and

$$\sup(0, x) \leqq \phi(x) \leqq \sup(0, x) + \epsilon.$$

Proof. Let $g(\tau)=e^{-1/\tau^2}$, $\tau\neq 0$ and $g(0)=0$. Let

$$f(x) = K\int_0^x\int_0^t g(\tau)g(\tau-\epsilon)d\tau dt,$$

where

$$K = \left(\int_0^\epsilon g(\tau)g(\tau-\epsilon)d\tau\right)^{-1}.$$

Then $f^{(n)}(0)=0, n=0, 1, \cdots, 0<f(\epsilon)<\epsilon, f'(\epsilon)=1$ and $f^{(n)}(\epsilon)=0, n=2, 3, \cdots$. Moreover $f''(x)>0$ for $0<x<\epsilon$. Denote $f(\epsilon)=\eta$ and

$$\begin{aligned}\phi(x) &= 0 & x &< \eta-\epsilon,\\ &= f(x+\epsilon-\eta) & \eta-\epsilon &\leqq x\leqq \eta,\\ &= x & \eta &< x.\end{aligned}$$

The function $\phi(x)$ then has the desired properties. Proof complete.

Since $\phi(x)$ is convex there is a support line at each point. Let $\psi(x, x_0) = ax+b$, $0 \leqq a \leqq 1$, denote the support function at $x = x_0$. Then $\psi(x_0, x_0) = \phi(x_0)$ and $\psi(x, x_0) \leqq \phi(x)$. Further let $(hs)^\infty$ denote the class of functions which have harmonic support and are infinitely differentiable in D.

LEMMA 2. *Suppose $v \in (hs)^\infty$ in D and ϵ is a positive number. Then there is a function $u \in (hs)^\infty$ in D such that* (1)

$$\sup(0, v) \leqq u \leqq \sup(0, v) + \epsilon.$$

Proof. Consider $u(z) = \phi[v(z)]$ and the family of functions $u(z, z_0) = \psi[v(z), v(z_0)]$, $z_0 \in D$. Since $u(z, z_0) = av(z) + b$, where $0 \leqq a$, we have $u(z, z_0) \in (hs)$. Moreover

$$u(z_0, z_0) = \psi[v(z_0), v(z_0)] = \phi[v(z_0)] = u(z_0),$$

and

$$u(z, z_0) = \psi[v(z), v(z_0)] \leqq \phi[v(z)] = u(z).$$

In view of Corollary 1.1, $u \in (hs)$. Applying Lemma 1 we see that u is infinitely differentiable so that $u \in (hs)^\infty$. Also

$$\sup[0, v(z)] \leqq \phi[v(z)] \leqq \sup[0, v(z)] + \epsilon$$

is then just the inequality of Lemma 2. Proof complete.

It may be remarked that the argument of the lemma is also valid for the following result. If $\phi(x)$ is increasing and convex over the range of values of an (hs)-function v then $u = \phi(v)$ is an (hs)-function. This corresponds to a theorem of Montel [8, p. 42] for subharmonic functions.

LEMMA 3. *Let $v = \sup(h_1, \cdots, h_n)$ where $h_1, \cdots, h_n$ are harmonic and single valued in D. Given any positive number ϵ, there exists a function $u \in (hs)^\infty$ in D such that $v \leqq u \leqq v + \epsilon$.*

Proof. The function $v_1 = h_1 - h_2$ belongs to $(hs)^\infty$ in D. Thus by Lemma 2, $\exists u_1 \in (hs)^\infty \ni$

$$\sup(v_1, 0) \leqq u_1 \leqq \sup(v_1, 0) + \epsilon/2.$$

By adding h_2 we obtain

$$\sup(h_1, h_2) \leqq u_1 + h_2 \leqq \sup(h_1, h_2) + \epsilon/2.$$

Suppose now that $u_{k-1} \in (hs)^\infty$ and

$$(3.1) \qquad \sup(h_1, \cdots, h_k) \leqq u_{k-1} + h_k \leqq \sup(h_1, \cdots, h_k) + \sum_{i=1}^{k-1} \epsilon/2^i.$$

(1) The author had originally obtained this result for $u \in (hs)^2$ with a longer proof. This more general result and concise proof was suggested by A. Brown.

Then the function $v_k = u_{k-1} + h_k - h_{k-1} \in (hs)^\infty$ in D. Again applying Lemma 2 $\exists u_k \in (hs)^\infty \ni$

$$\sup(v_k, 0) \leqq u_k \leqq \sup(v_k, 0) + \epsilon/2^k.$$

By adding h_{k+1} we obtain

$$\text{(3.2)} \quad \sup(u_{k-1} + h_k, h_{k+1}) \leqq u_k + h_{k+1} \leqq \sup(u_{k-1} + h_k, h_{k+1} + \epsilon/2^k.$$

The combination of inequalities (3.1) and (3.2) implies

$$\sup(h_1, \cdots, h_{k+1}) \leqq u_k + h_{k+1} \leqq \sup(h_1, \cdots, h_{k+1}) + \sum_{i=1}^{k} \epsilon/2^i.$$

Thus by induction on k we may assert $\exists u_{n-1} \in (hs)^\infty \ni$

$$\sup(h_1, \cdots, h_n) \leqq u_{n-1} + h_n \leqq \sup(h_1, \cdots, h_n) + \sum_{i=1}^{n-1} \epsilon/2^i$$

or setting $u = u_{n-1} + h_n$ we have $u \in (hs)^\infty$ and $v \leqq u \leqq v + \epsilon$. Proof complete.

THEOREM 5. *If $v \in (hs)$ in D, then there is a sequence of functions $\{u_n\}$ of class $(hs)^\infty$ which converges subuniformly to v in D.*

Let Δ denote a domain such that its closure $\Delta^- \subset D$. Let h denote a support function at z in Δ^- and choose a positive number ϵ. Then $\sup(v, h+\epsilon) = h+\epsilon$ in a neighborhood U of z and $v \leqq \sup(v, h+\epsilon) \leqq v+\epsilon$ in D. In view of the covering theorem there is a finite set of such neighborhoods which cover Δ^-. Denote the corresponding support functions $h_1, \cdots, h_n$. Then the function $v^* = \sup(v, h_1+\epsilon, \cdots, h_n+\epsilon)$ satisfies

$$v \leqq v^* \leqq v + \epsilon \text{ in } D$$

and

$$v^* = \sup(h_1 + \epsilon, \cdots, h_n + \epsilon) \text{ on } \Delta^-.$$

By Lemma 3 there is a function $u \in (hs)^\infty$ in $D \ni$

$$\sup(h_1 + \epsilon, \cdots, h_n + \epsilon) \leqq u \leqq \sup(h_1 + \epsilon, \cdots, h_n + \epsilon) + \epsilon.$$

Thus on Δ^-, $v \leqq u \leqq v^* + \epsilon \leqq v + 2\epsilon$. Now choose a sequence of domains Δ_k which exhaust D. Let $\{\epsilon_k\}$ be a sequence of positive numbers decreasing to zero and let $\{u_k\}$ be a sequence of functions in $(hs)^\infty$ such that in Δ_k^- $v \leqq u_k \leqq v + \epsilon_k$. Then $u_k \to v$ subuniformly in D. Proof complete.

4. **The distribution function.** The remarkable work of F. Riesz [13] n establishing the equivalence between the class of subharmonic functions and the class of logarithmic potentials of negative mass distributions initiated a very large literature on the theory and its applications. One might anticipate the possibility of characterizing the subclass of distribution functions corresponding to the class (hs). The following results lead to a necessary local

condition which must be satisfied by such a distribution. Since the class (hs) cannot be defined locally, however, any characterization of distribution functions would necessarily be in the large.

Consider a set function μ in a domain D. Suppose that the restriction of μ to a domain Δ, $\Delta^- \subset D$, is a non-negative (monotonic) additive set function on Borel sets in Δ as defined by Radon [11]. The function μ is called a generalized positive mass distribution by Radó [10, p. 42].

Suppose u is subharmonic in D. Then there is a unique generalized positive mass distribution μ in D such that for any domain Δ, $\Delta^- \subset D$

$$u(z) = \int_\Delta \log |z - \zeta| \, d\mu(e_\zeta) + h(z), \qquad z \in \Delta$$

where $h(z)$ is harmonic in Δ. Suppose further that $u \in (hs)$ and $u(z) < M$ in the circle $|z - z_0| < R$ which is in D. Let C_ρ denote the circle $|z - z_0| < \rho$, $\rho < R$. The purpose of this section is to show that $\mu(C_\rho) = O[\rho/(R^2 - \rho^2)]$.

To simplify the computations consider the function

$$u^*(\zeta) = \frac{u(R\zeta + z_0) - u(z_0)}{M - u(z_0)}.$$

The function $u^* \in (hs)$ in $|\zeta| < 1$ and satisfies there $u^*(\zeta) < 1$ and $u^*(0) = 0$. Actually $u^* \in (hs)$ in the image D^* of the domain D under the transformation $\zeta = (z - z_0)/R$. Let μ^* denote the set function in D^* defined as follows. If e, $e^- \subset D$, is a Borel set and e^* is its image in D^* then $\mu^*(e^*) = \mu(e)/[M - u(z_0)]$. Then $-\mu^*$ is the generalized mass distribution for u^*. In particular, denoting $r = \rho/R$, the circle C_r^*, $|\zeta| < r$, $r < 1$, is the image of C_ρ. It will be shown that $\mu^*(C_r^*) \leqq 2r/(1 - r^2)$ and thus for μ we have the result

$$\mu(C_\rho) \leqq [M - u(z_0)] \, R \frac{2\rho}{R^2 - \rho^2}, \qquad 0 < \rho < R.$$

Unless otherwise specified, in the remainder of this section let $u \in (hs)$ in $|z| < 1$ and satisfy there $u(z) < 1$ and $u(0) = 0$. Select a sequence $\{h_n\}$ of distinct support functions which contains a support function for each point of a dense set in $|z| < 1$. The sequence may be finite. It may be readily verified that $u = \sup_n h_n$. Or denoting $v_n = \sup(h_1, \cdots, h_n)$ we must have $v_n \to u$ subuniformly in $|z| < 1$, since v_n is a nondecreasing sequence with a continuous limit.

The idea of the demonstration utilizes Gauss' integral theorem. If γ is a sufficiently regular simple closed curve and u is the potential of a mass distribution $-\mu$ on a set E strictly interior to γ then

$$\int_\gamma \frac{\partial u}{\partial n} ds = 2\pi\mu(E),$$

(cf. Kellogg [7, p. 43]), where $\partial u/\partial n$ is the outward normal derivative on γ. Thus a bound on the normal derivative would lead to a bound on μ. In the case under consideration, the mass may be spread across the curve γ in which case $\partial u/\partial n$ may fail to exist. F. Riesz [13] defines what he calls best harmonic majorants of a subharmonic function and utilizes these to obtain an expression for the total mass inside γ as a limit of certain Gauss integrals. Brelot [3, p. 20] shows that Gauss' theorem is valid in general if u has continuous derivatives in a domain containing the curve.

Considering first functions of the form $v=\sup(h_1, \cdots, h_n)$ certain properties are obtained for the distribution functions. Utilizing the above results Gauss' theorem is shown to be valid for any circle. A bound is then obtained for $\partial v/\partial r$ by obtaining a bound for the normal derivative of the harmonic support functions. Having thus obtained a bound for the distribution in this case, the Riesz theory is again utilized to show that the result extends to the general function in (hs).

We now fix our attention on the circle C_ρ defined as $|z|<\rho$, $0<\rho<1$, and its boundary γ_ρ: $|z|=\rho$. The sets in C_ρ having nonzero mass for the function $v_2=\sup(h_1, h_2)$ are necessarily those which interest the curves $h_1-h_2=0$ since v_2 is harmonic at points off these curves. It is convenient to describe this situation by saying that the mass for v_2 is distributed on the curves $h_1-h_2=0$. The mass distribution for the function $v_3=\sup(h_1, h_2, h_3)$ is restricted to the curves $h_1-h_2=0$, $h_1-h_3=0$ and $h_2-h_3=0$. The distribution may very well be zero on portions of these curves. For example v_3 will be harmonic where h_3 exceeds h_1 and h_2 even though portions of the curves $h_1-h_2=0$ may be in such a region. In general the mass for the function $v_n=\sup(h_1, \cdots, h_n)$ will be distributed on the curves $h_i-h_j=0$, $i<j$. We will obtain certain properties of these distributions as a consequence of the two lemmas which follow.

First consider a simply connected domain D and the family of curves $h(z)=0$ where h is harmonic in D. These curves intersect only at the critical points $h_x=h_y=0$. By a branch of the family we will mean a curve, with all possible extensions in the region under consideration, which has a tangent at each point.

LEMMA 4. *In a neighborhood of each point of a branch of the curves $h(z)=0$ there is a parametric representation $x=\phi(v)$, $y=\psi(v)$ where ϕ and ψ are analytic. Moreover the representation is regular in the sense that $[\phi'(v)]^2+[\psi'(v)]^2\neq 0$.*

Proof. Let z_0 denote the point on the curve and choose the harmonic conjugate $k(z)$ of $h(z)$ such that $k(z_0)=0$. Then consider the function $f(z)=h(z)+ik(z)$. Suppose first that z_0 is not a critical point for $h(z)$, then $f'(z_0)\neq 0$. Thus $w=f(z)$ has a single valued inverse $z=f^{-1}(w)$ in a neighborhood of the origin such that $z_0=f^{-1}(0)$. The series representation $z=\sum_{n=0}^{\infty} a_n w^n$ is valid in a circle about the origin. The image of the curve $h(z)=0$ is the curve $w=iv$ so that, setting $a_n=\alpha_n+i\beta_n$, the original curve has the representation

$z(v)=\sum_{n=0}^{\infty}(\alpha_n+i\beta_n)(iv)^n=(\alpha_0-\beta_1 v-\alpha_2 v^2+\cdots)+i(\beta_0+\alpha_1 v-\beta_2 v^2+\cdots)$ $=\phi(v)+i\psi(v)$. On noting that $\phi'(0)=-\beta_1$ and $\psi'(0)=\alpha_1$ we have $[\phi'(0)]^2+[\psi'(0)]^2=\alpha_1^2+\beta_1^2=|a_1|^2=|1/f'(z_0)|^2\neq 0$. Thus in a suitable neighborhood of $v=0$, $x=\phi(v)$ and $y=\psi(v)$ give the desired representation.

Supposing now that z_0 is a critical point of $h(z)$ and specifically a zero of order n for $f(z)$, we may write $f(z)=(z-z_0)^n g(z)$, $g(z_0)\neq 0$. Denote $F(z)=[f(z)]^{1/n}=(z-z_0)[g(z)]^{1/n}=H(z)+iK(z)$ then $F(z)$ is holomorphic in a neighborhood of z_0 and

$$F'(z) = [g(z)]^{1/n} + (1/n)(z - z_0)[g(z)]^{1/n-1}$$

so that $[H_x(z_0)]^2+[H_y(z_0)]^2=|F'(z_0)|^2=|[g(z_0)]^{1/n}|^2\neq 0$. Thus z_0 is not a critical point for $H(z)$. We are interested then in the relation $f=h+ik=F^n=(H+iK)^n$.

Consider for a moment the function $z^n=r^ne^{in\theta}=r^n\cos n\theta+ir^n\sin n\theta$. $\mathcal{R}(z^n)=r^n\cos n\theta=0$ for $0=\pi/2n, 3\pi/2n, \cdots, (4n-1)\pi/2n$, or writing $z=x+iy$, $\mathcal{R}(z^n)=0$ on the lines $x+b_iy=0$, $i=1, 2, \cdots, n$, where the coefficients b_i depend on n. Specifically we may write $\mathcal{R}(z^n)=(x+b_1y)\cdot\cdots\cdot(x+b_ny)$.

Thus also $h=\mathcal{R}(F^n)=(H+b_1K)\cdot\cdots\cdot(H+b_nK)$ and the curves $h(z)=0$ through z_0 are described by the equations $H+b_iK=0$, $i=1,\cdots,n$. The function $\Phi_i=H+b_iK$ is harmonic in a neighborhood of z_0 and z_0 is not a critical point because

$$\partial\Phi_i/\partial x = H_x + b_iK_x = 0,$$
$$\partial\Phi_i/\partial y = H_y + b_iK_y = 0,$$

if and only if

$$\begin{vmatrix} H_x & K_x \\ H_y & K_y \end{vmatrix} = \begin{vmatrix} H_x & -H_y \\ H_y & H_x \end{vmatrix} = H_x^2 + H_y^2 = 0.$$

But $H_x^2+H_y^2\neq 0$ and we conclude $(\partial\Phi_i/\partial x)^2+(\partial\Phi_i/\partial y)^2\neq 0$ at z_0. Hence the curve $\Phi_i(z)=0$ has the analytic regular parametric representation of the lemma. Proof complete.

LEMMA 5. *Suppose* Γ_1 *and* Γ_2 *are curves which interpreted as plane sets are closed and which have an analytic regular parametric representation in a neighborhood of each point. If* Γ_1 *and* Γ_2 *are simple curves which intersect at a bounded infinite set of points, then they are identical along an arc.*

Proof. Let $\{z_n\}$ denote an infinite sequence of intersection points having limit point z_0. Using the closure property z_0 is also an intersection point. Let $x=\phi_1(t)$ and $y=\psi_1(t)$, $\alpha_1\leqq t\leqq\beta_1$, $x_0=\phi_1(t_0)$, $y_0=\psi_1(t_0)$ denote the parametric representation for Γ_1 in a neighborhood of $z_0=x_0+iy_0$. Note that in the case where z_0 is an end point the representation is one sided, i.e. $t_0=\alpha_1$ or $t_0=\beta_1$, but ϕ_1 and ψ_1 are still supposed analytic in a neighborhood $|t-t_0|<\delta$.

Similarly let $x=\phi_2(\tau)$ and $y=\psi_2(\tau)$, $\alpha_2 \leqq \tau \leqq \beta_2$, $x_0=\phi_2(\tau_0)$, $y_0=\psi_2(\tau_0)$ denote the parametric representation of Γ_2.

First, Γ_1 and Γ_2 must be tangent at z_0 for suppose the contrary. Using the fact that Γ_1 is simple we may restrict the curve to two arbitrarily small angular openings having the tangent as a bisector by choosing a sufficiently small neighborhood about z_0. Choose the angular opening to be less than the smallest angle between the two tangents. Make a similar restriction for the curve Γ_2. The smaller of the two neighborhoods then can contain no intersection points except z_0. This contradicts the fact that $\lim_n z_n = z_0$. Since the parametrization is regular, $\phi_1'(t_0) \neq 0$ or $\psi_1'(t_0) \neq 0$. Without loss of generality suppose $\phi_1'(t_0) \neq 0$, then also using the property of tangency $\phi_2'(\tau_0) \neq 0$.

Hence we may consider the inverse functions $t=\phi_1^{-1}(x)$ and $\tau=\phi_2^{-1}(x)$ which are analytic in a neighborhood of x_0. Then the functions $y_1(x) =\psi_1[\phi_1^{-1}(x)]$ and $y_2(x)=\psi_2[\phi_2^{-1}(x)]$ are analytic near x_0, but denoting $z_n=x_n+iy_n$ we have $y_1(x_n)=y_2(x_n)$ for $n>n_0$. The choice of n_0 need only insure that the point z_n is given by the representations chosen. We thus conclude that $y_1(x) \equiv y_2(x)$, and that Γ_1 and Γ_2 are identical along an arc. Proof complete.

It may be noted that the restriction to simple curves is not essential but is convenient and sufficient for the immediate purposes. The curves may be open or closed. We are now in a position to introduce the following ideas. By a continuation of a curve Γ_1 we will mean a curve Γ_2 which is identical with Γ_1 over some arc. For the class of curves under consideration a continuation is unique in the following sense. There can be no "*Y*" points. In other words there can be no point z_0 an interior point of Γ_1 and Γ_2 for which Γ_1 and Γ_2 are identical on one side and different on the other. This result follows from the argument in Lemma 5 on noting that in this case the two curves will be identical over an arc containing z_0 in its interior.

Suppose now that $h(z)$ is harmonic in a simply connected domain containing the closed disc $|z| \leqq \rho$. The curves $h(z)=0$ are necessarily simple in D for otherwise $h(z)\equiv 0$ by the maximum principle. Since the critical points are isolated there is at most a finite set in $|z| \leqq \rho$. If the associated holomorphic function has a zero of order n then there is a neighborhood of the point containing exactly $n+1$ branches of the family of curves. At other points the gradient is positive and there is a neighborhood of each containing exactly one branch. Choosing a finite covering of $|z| \leqq \rho$ one concludes that the family of curves is finite in $|z| \leqq \rho$. Furthermore the family forms a closed point set and hence each branch is closed. A branch cannot intersect $|z| = \rho$ in more than a finite set of points for then Lemma 5 and the remarks on continuation following it apply and the branch would itself be the closed curve $|z| = \rho$ which is impossible in view of the maximum principle.

Now apply these results to the family of curves on which the mass of $v_n = \max(h_1, \cdots, h_n)$ is distributed. Each of the $n(n-1)/2$ families of curves $h_i - h_j = 0$, $i < j$, has a finite set of branches and hence so does the entire family. New intersection points are introduced which no longer correspond to critical points but even these must form a finite set. An infinite set of intersections implies the same for a pair of branches and they must then be identical. Indeed one may eliminate from consideration all portions of each branch over which v_n is harmonic. There remains then a network of simple arcs each of which terminates at an intersection point or at the boundary $|z| = \rho$.

One may compare then the description of the surface v_n with the description of a convex polyhedron. The points above the intersection points are vertices, the curves above the simple arcs are edges and the components in which v_n is harmonic are the faces. Actually the convex polyhedron is a special case.

The essential results for our purpose at the moment is that v_n is harmonic on $|z| = \rho$ with the exception of a finite set of points and that the distribution network is closed in $|z| \leqq \rho$. We proceed using this information to prove the validity of Gauss' theorem for v_n. To accomplish this we return to the construction used to obtain an approximation in Lemmas 1, 2 and 3.

LEMMA 6. *Let $v = \sup(h_1, \cdots, h_n)$ where $h_1, \cdots, h_n$ are harmonic and single valued in a domain containing $|z| \leqq \rho$. If E is a closed set which does not intersect the distribution network for v then there is a function $u \in (hs)^\infty \ni u = v$ on E and*

$$\min\left(\frac{\partial h_1}{\partial r}, \cdots, \frac{\partial h_n}{\partial r}\right) \leqq \frac{\partial u}{\partial r} \leqq \max\left(\frac{\partial h_1}{\partial r}, \cdots, \frac{\partial h_n}{\partial r}\right)$$

in $0 < |z| \leqq \rho$.

Proof. We may suppose the functions $h_1, \cdots, h_n$ to be distinct. Decompose E into n sets $e_1, \cdots, e_n$ such that $v = h_i$ on e_i, $i = 1, \cdots, n$, then

$$h_i > \sup(h_1, \cdots, h_{i-1}, h_{i+1}, \cdots, h_n) \text{ on } e_i,\ i = 1, \cdots, n.$$

The strict inequality applies because e_i and the distribution network are closed sets and nonintersecting. There are positive constants m_1 and $M_1 \ni h_1 - h_2 \geqq m_1$ on e_1 and $h_1 - h_2 \leqq -M_1$ on e_2. Choosing a function ϕ_1 of Lemma 1 with suitably small ϵ we obtain $u_1 = \phi_1(h_1 - h_2) \in (hs)^\infty$ satisfying

$$u_1 = h_1 - h_2 \text{ on } e_1, \qquad u_1 = 0 \text{ on } e_2.$$

and

$$0 \leqq u_1 < h_3 - h_2 \text{ on } e_3.$$

There are positive constants m_2 and $M_2 \ni$

$$\begin{aligned} u_1 + h_2 - h_3 &= h_1 - h_3 \geqq m_2 \text{ on } e_1 \\ &= h_2 - h_3 \geqq m_2 \text{ on } e_2 \end{aligned}$$

and

$$u_1 + h_2 - h_3 \leqq - M_2 \text{ on } e_3.$$

For sufficiently small ϵ there is a function $\phi_2 \ni u_2 = \phi_2(u_1 + h_2 - h_3) \in (hs)^\infty$ and satisfies

$$u_2 = u_1 + h_2 - h_3 \text{ on } e_1 \cup e_2,$$
$$u_2 = 0 \text{ on } e_3$$

and

$$0 \leqq u_2 < h_4 - h_3 \text{ on } e_4.$$

To obtain a sequence we proceed by induction. Suppose $u_j \in (hs)^\infty$ and satisfies

$$u_j = u_{j-1} + h_j - h_{j+1} \text{ on } \bigcup_{i=1}^{j} e_i,$$
$$u_j = 0 \text{ on } e_{j+1}$$

and

$$0 \leqq u_j < h_{j+2} - h_{j+1} \text{ on } e_{j+2}, \qquad j = 2, \cdots, k,\ 2 \leqq k \leqq n - 3.$$

Then there exist positive constants m_{k+1} and $M_{k+1} \ni$

$$u_k + h_{k+1} - h_{k+2} = h_i - h_{k+2} \geqq m_{k+1} \text{ on } e_i, \qquad i = 1, \cdots, k+1$$

and

$$u_k + h_{k+1} - h_{k+2} \leqq - M_{k+1} \text{ on } e_{k+2}.$$

Thus there is a function $\phi_{k+1} \ni u_{k+1} = \phi_{k+1}(u_k + h_{k+1} - h_{k+2}) \in (hs)^\infty$ and satisfies

$$u_{k+1} = u_k + h_{k+1} - h_{k+2} \text{ on } \bigcup_{i=1}^{k+1} e_i,$$
$$u_{k+1} = 0 \quad \text{on } e_{k+2}$$

and

$$0 \leqq u_{k+1} < h_{k+3} - h_{k+2} \text{ on } e_{k+3}.$$

Hence we obtain the sequence $u_1, \cdots, u_{n-2}$ with the above properties. Repeating the argument for the function $u_{n-2} + h_{n-1} - h_n$ we obtain a function $u_{n-1} = \phi_{n-1}(u_{n-2} + h_{n-1} - h_n) \in (hs)^\infty$ and satisfying

$$u_{n-1} = u_{n-2} + h_{n-1} - h_n \text{ on } \bigcup_{i=1}^{n-1} e_i,$$

and

$$u_{n-1} = 0 \text{ on } e_n.$$

The function $u=u_{n-1}+h_n$ then takes the values $u=h_i$ on e_i or $u=v$ on the set E.

To obtain the inequality for the radial derivative notice that $0\leqq\phi_i'(x)\leqq 1$ for all x and $i=1,\cdots,n-1$. For convenience denote $\alpha_k=\phi_k'(u_{k-1}+h_k-h_{k+1})$, $k=1,\cdots,n-1$ where $u_0=0$. Observing that

$$\frac{\partial u_k}{\partial r}=\alpha_k\left(\frac{\partial u_{k-1}}{\partial r}+\frac{\partial h_k}{\partial r}-\frac{\partial h_{k+1}}{\partial r}\right),$$

the formula

$$\frac{\partial u_k}{\partial r}=\alpha_k\cdot\cdots\cdot\alpha_1\frac{\partial h_1}{\partial r}+\alpha_k\cdot\cdots\cdot\alpha_2(1-\alpha_1)\frac{\partial h_2}{\partial r}+\cdots$$
$$+\alpha_k(1-\alpha_{k-1})\frac{\partial h_k}{\partial r}-\alpha_k\frac{\partial h_{k+1}}{\partial r},\qquad k=1,\cdots,n-1$$

may be established by induction. Then

$$\frac{\partial u}{\partial r}=\alpha_{n-1}\cdot\cdots\cdot\alpha_1\frac{\partial h_1}{\partial r}+\alpha_{n-1}\cdot\cdots\cdot\alpha_2(1-\alpha_1)\frac{\partial h_2}{\partial r}+\cdots$$
$$+\alpha_{n-1}(1-\alpha_{n-2})\frac{\partial h_{n-1}}{\partial r}+(1-\alpha_{n-1})\frac{\partial h_n}{\partial r}.$$

Consider the expression

$$y=\alpha_{n-1}\cdot\cdots\cdot\alpha_1x_1+\alpha_{n-1}\cdot\cdots\cdot\alpha_2(1-\alpha_1)x_2+\cdots$$
$$+\alpha_{n-1}(1-\alpha_{n-2})x_{n-1}+(1-\alpha_{n-1})x_n$$

in the variables α_i, $0\leqq\alpha_i\leqq 1$. It is linear in each α_i and thus will take its maximum with respect to α_i for $\alpha_i=0$ or $\alpha_i=1$. The maximum value for the expression will be taken then for values 0 or 1 for all the variables. Suppose j is the largest index such that $\alpha_j=0$. Then the coefficients of $x_1,\cdots,x_j$, $x_{j+2},\cdots,x_n$ are zero and $y=x_{j+1}$. The same argument applies for the minimum value so that $\min(x_1,\cdots,x_n)\leqq y\leqq\max(x_1,\cdots,x_n)$. The inequality of the lemma for $\partial u/\partial r$ is thus valid in $0<|z|\leqq\rho$. Proof complete.

LEMMA 7. *Let $v=\sup(h_1,\cdots,h_n)$ where $h_1,\cdots,h_n$ are harmonic and single valued in a domain containing $|z|\leqq\rho$. If $-\nu$ denotes the mass distribution for v, then*

$$\int_{\gamma_\rho}\frac{\partial v}{\partial r}ds=2\pi\nu(C_\rho).$$

Proof. Given $\epsilon>0$ choose a closed set E as follows. Choose a closed set E_ρ on γ_ρ whose angular measure at the origin is more than $2\pi-\epsilon$ and which contains no points of the mass distribution network for v. Choose $\rho_1<\rho\ni$ the portion of the rays through the origin and points of E_ρ which lie in the annulus $\rho_1\leqq|z|\leqq\rho$ do not intersect the mass distribution network for v. Let E denote the union of these intervals. By Lemma 6 choose $u\in(hs)^\infty$ such that $u=v$ on E and

$$\min\left(\frac{\partial h_1}{\partial r},\cdots,\frac{\partial h_n}{\partial r}\right)\leqq\frac{\partial u}{\partial r}\leqq\max\left(\frac{\partial h_1}{\partial r},\cdots,\frac{\partial h_n}{\partial r}\right)$$

or

$$\left|\frac{\partial u}{\partial r}\right|\leqq\max\left(\left|\frac{\partial h_1}{\partial r}\right|,\cdots,\left|\frac{\partial h_n}{\partial r}\right|\right)\leqq K\text{ on }0<|z|\leqq\rho.$$

Now $\partial v/\partial r$ is piecewise continuous on $|z|=r$, $r\leqq\rho$, and

$$\left|\frac{\partial v}{\partial r}\right|\leqq\max\left(\left|\frac{\partial h_1}{\partial r}\right|,\cdots,\left|\frac{\partial h_n}{\partial r}\right|\right)\leqq K\text{ on }0<|z|\leqq\rho$$

wherever the derivative exists. Hence $\int_{\gamma_r}(\partial v/\partial r)ds$ exists for each r, $0<r\leqq\rho$. Moreover setting $E_r=\gamma_r\cap E$

$$\begin{aligned}\left|\int_{\gamma_r}\frac{\partial v}{\partial r}ds-\int_{\gamma_r}\frac{\partial u}{\partial r}ds\right|&\\ &=\left|\int_{E_r}\left(\frac{\partial v}{\partial r}-\frac{\partial u}{\partial r}\right)ds+\int_{\gamma_r-E_r}\frac{\partial v}{\partial r}ds-\int_{\gamma_r-E_r}\frac{\partial u}{\partial r}ds\right|\\ &\leqq\int_{\gamma_r-E_r}\left|\frac{\partial v}{\partial r}\right|ds+\int_{\gamma_r-E_r}\left|\frac{\partial u}{\partial r}\right|ds\leqq\epsilon 2\rho K,\qquad\rho_1\leqq r\leqq\rho,\end{aligned}$$

where K is independent of r. Thus a sequence u_k, $k=1,2,\cdots$ may be chosen such that $\int_{\gamma_r}(\partial u_k/\partial r)ds$ converges uniformly to $\int_{\gamma_r}(\partial v/\partial r)ds$ with respect to r in the interval $\rho_1\leqq r\leqq\rho$. Noting that each integral $\int_{\gamma_r}(\partial u_k/\partial r)ds$ is a continuous function of r we conclude that $\int_{\gamma_r}(\partial v/\partial r)ds$ is a continuous function of r. Also the functions u_k have continuous derivatives and using the result of Brelot, if $-\mu_k$ denotes the corresponding mass distribution, then

$$\int_{\gamma_\rho}(\partial u_k/\partial r)ds=2\pi\mu_k(C_\rho).$$

In the special case of the function u_k in the annulus $\rho_1\leqq|z|\leqq\rho$ the best harmonic majorant is just the function w_k harmonic in $\rho_1<|z|<\rho$ and taking the values of u_k on the boundary $|z|=\rho_1$ and $|z|=\rho$. Then if $\rho_1<r<\rho$,

$$2\pi\mu_k(C_{\rho_1})\leqq\int_{\gamma_r}\frac{\partial w_k}{\partial r}ds\leqq 2\pi\mu_k(C_\rho)$$

whence

$$\int_{\gamma_{\rho_1}} \frac{\partial u_k}{\partial r}\, ds \leqq \int_{\gamma_r} \frac{\partial w_k}{\partial r}\, ds \leqq \int_{\gamma_\rho} \frac{\partial u_k}{\partial r}\, ds. \tag{4.1}$$

If w is the function harmonic in $\rho_1 < |z| < \rho$ and taking the boundary values of v, then

$$\int_{\gamma_r} \frac{\partial w}{\partial r}\, ds \to 2\pi\nu(C_\rho) \text{ as } \rho_1 \to \rho. \tag{4.2}$$

Since the sequence $u_k \to v$ uniformly on $\rho_1 \leqq |z| \leqq \rho$ so must $w_k \to w$ and $\partial w_k/\partial r \to \partial w/\partial r$ uniformly on $|z| = r$. Hence

$$\int_{\gamma_r} \frac{\partial w_k}{\partial r}\, ds \to \int_{\gamma_r} \frac{\partial w}{\partial r}\, ds \text{ as } k \to \infty$$

and taking the limit of (4.1) as $k \to \infty$ we obtain

$$\int_{\gamma_{\rho_1}} \frac{dv}{\partial r}\, ds \leqq \int_{\gamma_r} \frac{\partial w}{\partial r}\, ds \leqq \int_{\gamma_\rho} \frac{\partial v}{\partial r}\, ds.$$

Taking now the limit as $\rho_1 \to \rho$, using (4.2) and the continuity of the integrals we have

$$\int_{\gamma_\rho} \frac{\partial v}{\partial r}\, ds = 2\pi\nu(C_\rho).$$

Proof complete.

Having obtained the Gauss' integral for computing the mass $\nu(C_\rho)$ in terms of $\partial v/\partial r$ we proceed to determine a bound for the latter function. Its value where defined is just the value of the derivative of the appropriate harmonic support function. Thus we need only consider the corresponding problem for a harmonic function under restrictions imposed by Harnack's inequality. We have $v < 1$ and any support function $h < 1$ and applying the inequality to $h - 1$ there results

$$\frac{[2 - h(0)]\,|z| - h(0)}{|z| - 1} \leqq h(z) \leqq \frac{[2 - h(0)]\,|z| + h(0)}{|z| + 1}. \tag{4.3}$$

This inequality will be utilized to prove the next lemma.

LEMMA 8. *If $v = \sup(h_1, \cdots, h_n)$, where $h_1, \cdots, h_n$ are harmonic and bounded by 1 in $|z| < 1$, and has distribution function $-\nu$, then for $0 < \rho < 1$*

$$\nu(C_\rho) \leqq \frac{2\rho[1 - v(0)]}{1 - \rho^2}.$$

Proof. Suppose $\partial v/\partial r$ is defined at $z=\rho$ and let h denote its support there. Consider the mapping $z=(w+\rho)/(1+\rho w)$. The function

$$\phi(w) = h\left(\frac{w+\rho}{1+\rho w}\right)$$

is harmonic and $\phi(w)<1$ in $|w|<1$. Also $\phi(0)=h(\rho)=v(\rho)$. Denoting $\phi(0)=\alpha$ we have from (4.3),

$$\phi(w) \leqq \frac{(2-\alpha)|w| + \alpha}{|w| + 1}$$

and substituting $w=(z-\rho)/(1-\rho z)$

$$h(z) \leqq \frac{(2-\alpha)|(z-\rho)/(1-\rho z)| + \alpha}{|(z-\rho)/(1-\rho z)| + 1} \qquad \text{in } |z| < 1.$$

In particular for $z=r$, $\rho\leqq r<1$,

$$h(r) \leqq \frac{2(r-\rho)+\alpha(1+\rho)(1-r)}{(1-\rho)(1+r)} = \psi(r).$$

Since $h(\rho)=\psi(\rho)=\alpha$ it follows that

$$\frac{\partial v}{\partial r} = \frac{\partial h}{\partial r} \leqq \psi'(\rho) = \frac{2(1-\alpha)}{1-\rho^2} \qquad \text{at } r=\rho.$$

Also at $z=\rho e^{i\theta}$

$$\frac{\partial v}{\partial r} \leqq \frac{2[1-v(\rho e^{i\theta})]}{1-\rho^2}$$

wherever $\partial v/\partial r$ exists. Now letting h denote the support at $z=0$ we have $v(\rho e^{i\theta})\geqq h(\rho e^{i\theta})$ giving

$$\frac{\partial v}{\partial r} \leqq \frac{2[1-h(\rho e^{i\theta})]}{1-\rho^2}.$$

In view of Lemma 7

$$\nu(C_\rho) = \frac{1}{2\pi}\int_{\gamma_\rho} \frac{\partial v}{\partial r}\, ds \leqq \frac{2\rho}{1-\rho^2}\left[1 - \frac{1}{2\pi}\int_0^{2\pi} h(\rho e^{i\theta})d\theta\right].$$

The last integral is just the mean value of h on $|z|=\rho$. Since $h(0)=v(0)$ there results.

$$\nu(C_\rho) \leqq \frac{2\rho[1-v(0)]}{1-\rho^2}.$$

Proof complete.

The Riesz theory may again be applied to obtain the following general result.

THEOREM 6. *Suppose* $u \in (hs)$ *in a domain* D *containing* $|z|<1$, $u<1$ *in* $|z|<1$ *and* $u(0)=0$. *If* $-\mu$ *is the distribution function and* $0<\rho<1$ *then* $\mu(C_\rho) \leqq 2\rho/(1-\rho^2)$.

Proof. Let $\{h_n\}$ denote a sequence of support functions at a dense set of points in $|z| \leqq \rho$. Then $v_n = \sup(h_1, \cdots, h_n)$ increases uniformly to u on $|z| \leqq \rho$. Let h and h_n denote respectively the best harmonic majorants of u and v_n on $\rho_1 \leqq |z| \leqq \rho$, $0<\rho_1<\rho$, then

$$2\pi\mu(C_{\rho_1}) \leqq \int_{\gamma_r} \frac{\partial h}{\partial r}\, ds \leqq 2\pi\mu(C_\rho),$$

and

$$2\pi\nu_n(C_{\rho_1}) \leqq \int_{\gamma_r} \frac{\partial h_n}{\partial r}\, ds \leqq 2\pi\nu_n(C_\rho),$$

where $-\nu_n$ is the distribution function for v_n and $\rho_1<r<\rho$. The uniform convergence of v_n to u on the annulus implies $\partial h_n/\partial r \to \partial h/\partial r$ uniformly on $|z|=r$ and thus

$$\int_{\gamma_r} \frac{\partial h_n}{\partial r}\, ds \to \int_{\gamma_r} \frac{\partial h}{\partial r}\, ds.$$

These results lead to the inequality

$$\liminf_{n\to\infty} \nu_n(C_\rho) \geqq \mu(C_{\rho_1})$$

but since $\mu(C_{\rho_1}) \to \mu(C_\rho)$ as $\rho_1 \to \rho$ we obtain

$$\liminf_{n\to\infty} \nu_n(C_\rho) \geqq \mu(C_\rho).$$

Applying Lemma 8

$$\nu_n(C_\rho) \leqq \frac{2\rho[1 - v_n(0)]}{1 - \rho^2},$$

and the fact that $v_n(0) \to 0$, the inequality

$$\mu(C_\rho) \leqq \frac{2\rho}{1 - \rho^2}$$

is obtained. Proof complete.

If $\rho<\rho_2<1$ then $\mu(C_{\rho_2}) \to \mu(C_\rho \cup \gamma_\rho)$ as $\rho_2 \to \rho$ so that

$$\mu(C_\rho \cup \gamma_\rho) \leqq \frac{2\rho}{1-\rho^2}\,.$$

The following example shows that the inequality of the theorem is sharp except perhaps for the omission of the equality sign(2). The function

$$h(z) = \mathfrak{R}\left(\frac{2(\rho - z)}{(\rho - 1)(1 + z)}\right)$$

has radial derivative $\partial h/\partial r = 2(1+\rho)/(1-\rho)(1+r)^2$ at $z=r>0$, and $h(z)<h(r)$ on $|z|=r$, $z\neq r$. Let u denote the supremum of zero and all rotations of $h(z)$, i.e. $h(ze^{i\theta})$, $0\leqq\theta<2\pi$. Then for $\rho<|z|<1$, u has continuous derivatives and $\mu(C_r) = 2\rho(1+\rho)/(1-\rho)(1+r)^2, \rho<r<1$. As $r\to\rho\; \mu(C_r)\to 2\rho/(1-\rho^2)$ or $\mu(C_\rho \cup \gamma_\rho) = 2\rho/(1-\rho^2)$.

The bound is exceeded by distributions for subharmonic functions as is illustrated by the example $v=|z|^\alpha$, $0<\alpha<1$, which has distribution $\alpha\rho^\alpha$ on C_ρ.

Carrying out the preceding development for a convex function $c(z)$ satisfying $c(z)<1$ in $|z|<1$ and $c(0)=0$ a corresponding bound is obtained for its distribution function γ, namely,

$$\gamma(C_\rho) \leqq \frac{\rho}{1-\rho}\,.$$

The above example for $\alpha\geqq 1$ is such a function.

5. **Normality.** As promised in §3 the bound on the distribution function will be utilized to show that a family which is subuniformly bounded above is normal. The proof is simply an application of a theorem of Arsove [1].

THEOREM 7. *Let $\mathfrak{U}$ denote a family of functions $u\in(hs)$ in a domain D. If $\mathfrak{U}$ is subuniformly bounded above in D then it is normal.*

Proof. It is sufficient to show that $\mathfrak{U}$ is normal in each circle C: $|z-z_0|<R$ in D. Let $C_\rho(\zeta)$ denote the circle $|z-\zeta|<\rho$, $\rho\leqq d$, where d is less than the distance between the boundaries of C and D. Then there exists a constant $M\ni$ each $u\in\mathfrak{U}$ satisfies $u<M$ for $z\in C_d(\zeta)$, $\zeta\in C$. Consider a sequence $\{u_n\}$ from the family $\mathfrak{U}$. The sequence $u_n(z_0)$ is bounded or $u_n(z_0)\to-\infty$. Application of the Harnack inequality (Theorem 2) shows that the sequence $\{u_n\}$ is uniformly bounded or tends to $-\infty$ uniformly in C. Considering the bounded case let $-\mu_n$ denote the mass distribution for u_n. Then for $0<\rho<d$

$$\mu_n[C_\rho(\zeta)] \leqq [M - u_n(\zeta)]\,\frac{2\rho d}{d^2-\rho^2}, \qquad \zeta\in C.$$

(2) *Added in proof.* Equality is impossible.

Letting m denote the lower bound for the sequence $\{u_n\}$ in C

$$\mu_n[C_\rho(\zeta)] \leqq (M - m)\frac{2\rho d}{d^2 - \rho^2}.$$

Consider

$$\psi_n(r) = \sup_{\zeta\in C} \int_0^r \frac{\mu_n[C_\rho(\zeta)]}{\rho}\, d\rho,$$

$$\leqq (M - m)2d\int_0^r \frac{d\rho}{d^2 - \rho^2} = (M - m)\log\frac{d+r}{d-r}, \quad 0 \leqq r < d,$$

and $\Psi(r) = \sup_n \psi_n(r) \leqq (M-m)\log(d+r)/(d-r)$. $\Psi(r) \geqq 0$ and the result $\lim_{r\to 0}\Psi(r) = 0$ implies the bounded sequence $\{u_n\}$ is normal in C. Consequently the family $\mathfrak{U}$ is normal in C. Proof complete.

6. **Subclasses of** (hs). The theorem to follow leads to interesting subclasses of (hs).

THEOREM 8. *If each function $v_1, \cdots, v_n$ is a nonnegative (hs)-function in D then for $\alpha > 1$, $v = (v_1^\alpha + \cdots + v_n^\alpha)^{1/\alpha} \in (hs)$.*

Proof. If $z_0 \in D$ and $v(z_0) = 0$ then $h(z) \equiv 0$ is a support function. If $v(z_0) > 0$, then

$$v^*(z) = \left(\frac{v_1(z_0)}{v(z_0)}\right)^{\alpha-1} v_1(z) + \cdots + \left(\frac{v_n(z_0)}{v(z_0)}\right)^{\alpha-1} v_n(z)$$

satisfies $v^* \in (hs)$, $v^*(z_0) = v(z_0)$ and $v^*(z) \leqq v(z)$ in D, the latter result being a form of Holder's inequality. Thus $v \in (hs)$ in D by Corollary 1.1. Proof complete.

If v is harmonic in D then $|v| = \sup(v, -v) \in (hs)$. Thus $|f| = (u^2+v^2)^{1/2}$, where $f = u + iv$ is holomorphic in D, belongs to (hs) in D. Also if $\mathfrak{F}$ is a subuniformly bounded family of function f holomorphic in D then $v = \sup_f |f|$ is an (hs)-function in D.

DEFINITION 2. A function v belongs to the class (pl) in D if and only if $\log v$ belongs to (hs) in D.

The corresponding class with respect to subharmonic functions has been considered by Radó, Privaloff, Beckenbach, Riesz and others. Suppose $\mathfrak{F}$ is a subuniformly bounded family of functions f holomorphic and nonzero in D, then $v = \sup_f |f| \in (pl)$. For $\log v = \sup_f \log |f|$ where the functions $\log |f|$ are harmonic in D.

If D is simply connected and $v \in (pl)$ then $\exists$ a family $\mathfrak{F}$ of functions f holomorphic in $D \ni v = \sup_f |f|$. Let h denote a harmonic support for $\log v$ at z_0. Let $f = \exp(h + ik)$ where k is a harmonic conjugate of h. Then $|f(z_0)| = \exp h(z_0) = \exp \log v(z_0) = v(z_0)$, $|f(z)| = \exp h(z) \leqq \exp \log v(z)$ in D and f is holomorphic in D. Hence $v = \sup_f |f|$. In particular we may state the result.

THEOREM 9. *If D is simply connected then a necessary and sufficient condition that $v \in (pl)$ in D is that there exist a family of functions f holomorphic and nonzero in D such the $v = \sup_f |f|$.*

Note that if D is simply connected then $(pl) \subset (hs)$. This is not true if D is multiply connected. For example $v = |z|^{1/2}$ is in (pl) in $0 < |z| < 1$ but not in (hs). It may happen that v is in (pl) and (hs) but not the supremum of the moduli of a family of holomorphic functions. For example $v = |z|^{\alpha}$, $0 < |z| < 1$, where $\alpha > 1$ but not an integer, is in (hs) and its logarithm is harmonic and single valued. If g is analytic in $0 < |z| < 1$ and $|g|$ supports v at z_0 then $\log |g(z_0)| = \alpha \log |z_0|$ and $\log |g(z)| \leqq \alpha \log |z|$ where both functions are harmonic in a neighborhood of z_0. Application of the maximum principle implies they are equal so that $g = e^{i\theta} z^{\alpha}$ which is not holomorphic.

In summary if we denote the class of subharmonic functions by (sh) and the class of supremums of the moduli of holomorphic functions by (ms) then in any domain $D(sh) \supset (hs) \supset (ms)$. If D is simply connected then also $(ms) \supset (pl)$. All of the inclusions are proper.

Another subclass of the functions of Theorem 8 is that of functions of the form $(u^2 + v^2 + w^2)^{1/2}$, where u, v and w are a triple of conjugate harmonic functions, studied by Beckenbach and Radó [2].

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